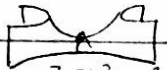
Lesson 23: Extrema of Functions of Two Variables

Relative Extrema: $f(x,y)$ has a local maximum at (a,b) if $f(a,b)$ is larger than $f(x,y)$ when (x,y) is near (a,b).
$f(x,y)$ has a local minimum at (a,b) if $f(a,b)$ is smaller than $f(x,y)$ when (x,y) is near (a,b).

Critical Point: A point (a,b) where $f_x(a,b)=0$ and $f_y(a,b)=0$. (OR DNE)

Notice: All relative extrema occur at critical points.
BUT not every critical point gives us a maximum or minimum.
Ex:  is called a saddle point, and is neither a max nor min.
$z = y^2 - x^2$

Second Derivative Test: Find all critical points (a,b) (set $f_x=0$, $f_y=0$). Let $D = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local min (conc. up)
If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local max (conc. down)
If $D<0$, then (a,b) is a saddle point. (If $D=0$, the test fails).

Ex 1: $f(x,y) = x^2 - y^2$

$f_x = 2x \overset{set}{=} 0 \rightarrow x=0$
$f_y = -2y \overset{set}{=} 0 \rightarrow y=0$
$f_{xx} = 2$
$f_{yy} = -2$, $f_{xy} = 0$ $D = 2(-2) - 0^2$

Point(s)	D(a,b)	$f_{xx}(a,b)$	Extrema
(0,0)	$2(-2)-0<0$		saddle

Ex 2: $f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x = 4x^3 - 4y \overset{set}{=} 0 \rightarrow y = x^3$
$f_y = 4y^3 - 4x \overset{set}{=} 0 \xrightarrow[y=x^3]{sub} 4x^9 - 4x = 0$
$f_{xx} = 12x^2$ $4x(x^8 - 1) = 0$
$f_{yy} = 12y^2$ $x=0$ (so $y=0^3=0$); $x=1$ (so $y=1^3=1$); $x=-1$ (so $y=(-1)^3=1$)
$f_{xy} = -4$ $D = 12x^2 \cdot 12y^2 - (-4)^2$

Point(s)	D(a,b)	f_{xx}	Extrema
(0,0)	$-16<0$		saddle
(1,1)	$128>0$	$12>0$	relative min
(-1,-1)	$128>0$	$12>0$	relative min

Ex 3: $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$

$f_x = -2 - 2x \overset{set}{=} 0 \rightarrow x = -1$
$f_y = 4 - 8y \overset{set}{=} 0 \rightarrow y = \frac{1}{2}$
$f_{xx} = -2$ $f_{xy} = 0$ $D = -2(-8) - 0^2$
$f_{yy} = -8$

Point(s)	D(a,b)	f_{xx}	Extrema
$(-1, 1/2)$	$16>0$	$-2<0$	relative max